

# ON NASH'S 4-SPHERE AND PROPERTY 2R

Motoo Tange

## Abstract

D.Nash defined a family of homotopy 4-spheres in [11]. Proving that his manifolds  $\mathcal{S}_{m,n,m',n'}$  are all real  $S^4$ , we find that they have handle decomposition with no 1-handles, two 2-handles and two 3-handles. The handle structures give new potential counterexamples of Property 2R conjecture.

## 1 Introduction

The smooth Poincaré conjecture in 4-dimension is still open. Though many people [3, 4] have proposed potential counterexamples, what some of them are standard  $S^4$  was proved [1, 7, 8, 9]. D. Nash [11] also proposed potential counterexamples of the conjecture. Most recently S.Akbulut [2] proved that the manifolds are all standard. In the article I will also give an alternative proof and furthermore remark some handle decompositions appeared there.

Nash's manifolds are constructed by log transformations along four tori in some 4-manifold. Hence we will give a brief review of the surgery. For the remark of the handle decomposition as stated above we introduce notions; Property nR, generalized Property R.

### 1.1 Log transformation.

Here we review the notation of the log transformation. Let  $T \subset X^4$  be a torus embedding with the trivial normal bundle  $\nu(T) = D^2 \times T$  in 4-manifold  $X$ . Removing the neighborhood, we reglue it with the map  $\varphi : \partial D^2 \times T^2 \rightarrow \partial \nu(T)$  satisfying

$$\varphi(\partial D^2 \times \{\text{pt}\}) = p\mu + q\gamma,$$

where  $\mu$  is the meridian of  $T$  and  $[\mu]$  is a primitive element of  $H_1(T)$ , so that we obtain a manifold.

**Definition 1.** *The surgery whose gluing map is  $\varphi$  as above*

$$X - \nu(T) \cup_{\varphi} (D^2 \times T)$$

*is called the  $(p/q)$ -log transformation along  $T$  with direction  $\gamma$ .*

### 1.2 Generalized Property R Conjecture.

Property R conjecture was proved by Gabai [6]. M. Scharlemann and A. Thompson in [12] generalized Property R as follows.

**Definition 2** ([12]). *We say that a knot  $K$  has Property nR if  $K$  satisfies the following property. If any  $n$ -component link  $L$  containing  $K$  as a component yields  $\#^n S^1 \times S^2$  by an integral Dehn surgery, then after some handle slidings the framed link can be reduced to the  $n$ -component unlink.*

The case where  $n = 1$  is equivalent to original Property R.

**Conjecture 1** (Generalized Property R Conjecture). *All knot admit Property  $nR$  for any  $n \geq 1$ .*

The generalized Property R conjecture is still open. The homotopy 4-spheres by D.Nash in [11] are standard, however we show that diagrams coming from handle decompositions might be counterexamples of the generalized Property R conjecture.

We can find Figure 1 along the way of proof that Nash's manifolds are standard (Theorem 1). The framed link with black color is a presentation of  $S^3$ . Each box stands

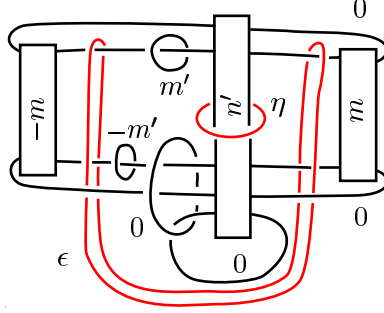


Figure 1: Examples which might not have Property 2R.

for the full twist by the number in the box. The 0-surgery along the 2-component link (red color) gives rise to  $\#^2 S^2 \times S^1$ , because the framed link gives a handle diagram of Nash's homotopy 4-sphere, which are indeed real  $S^4$  (Corollary 2).

**Question 1.** *Are  $\eta$  and  $\epsilon$  in Figure 1 examples not having Property 2R for any non-zero integers  $m, n, m'$ ?*

## Acknowledgements

The whole work of the paper was done during my visiting of The University of Texas at Austin in 2011. I deeply appreciate Professor Cameron Gordon's acceptance of my visiting, his, Professor John Luecke's and Dr. Cagri Karakurt's hospitality, many pieces of chalk and wide blackboard to draw some pictures. Also, this research was supported by Grant-in-Aid for JSPS Fellow(21-1458).

## 2 Nash's manifolds.

D. Nash in [11] defined a new family of homotopy 4-spheres as follows. Let  $A$  be a 4-manifold with the handle diagram Figure 2.

Since  $A$ , as in [5], is constructed by attaching two 2-handles to  $D^2 \times T_0^2 \subset (D^2 \times S^1) \times S^1$  along Bing tori of  $D^2 \times S^1$ ,  $A$  is also included a Bing tori  $B_T$  as in Figure 3, where  $T_0^2$  is the punctured torus. As a fundamental fact 0-surgery along  $B_T$  yields  $T_0^2 \times T_0^2$ . Here 0-surgery means  $(0/1)$ -log transformation. In addition two tori reglued by the surgery are  $T_1 = S_\alpha^1 \times S_\gamma^1$  and  $T_2 = S_\beta^1 \times S_\gamma^1$ , where  $S_\alpha^1, S_\beta^1, S_\gamma^1$  are generating circles of  $T_0^2 \subset T^2$ . In other words tori  $(0/1)$ -surgery along  $T_1^2$  and  $T_2^2$  yield  $A$  back.

Now we take two copies of  $T_0^2 \times T_0^2$  to glue the boundaries  $S^1 \times T_0^2 \cup T_0^2 \times S^1$  each other by gluing map  $\phi : S^1 \times T_0^2 \cup T_0^2 \times S^1 \rightarrow S^1 \times T_0^2 \cup T_0^2 \times S^1$  such that the two components are exchanged. We call the resulting manifold  $X$ . Such a construction

is generalized to  $n$ -component case by Fintushel-Stern in [5], and it is called *pinwheel construction*. Here we define the  $(m/1)$ -surgery of  $T_0^2 \times T_0^2$  along  $T_1$  with direction  $S_\alpha^1$  and simultaneously the  $(n/1)$ -surgery along  $T_2$  with direction  $S_\beta^1$  to be  $X_{m,n}$ . We define, by the same gluing map  $\phi$ ,  $X_{m,n} \cup_\phi X_{m',n'}$  to be  $\mathcal{S}_{m,n,m',n'}$ . From the construction immediately we have the following.

**Lemma 1.** *For any integer  $m, n, m', n'$  we have the following diffeomorphism*

$$\mathcal{S}_{m,n,m',n'} \cong \mathcal{S}_{m',n',m,n}.$$

Here Nash got a result (Theorem 3.2 in [11]), which the manifolds  $\mathcal{S}_{m,n,m',n'}$  are all homotopy 4-spheres. Namely  $\mathcal{S}_{m,n,m',n'}$  are candidates of the 4-dimensional smooth Poincaré conjecture. Are the manifolds diffeomorphic to standard  $S^4$ ? Here we give an affirmative answer for the question.

**Theorem 1** (Nash's manifolds are standard.). *The manifolds  $\mathcal{S}_{m,n,m',n'}$  are all diffeomorphic to the standard 4-sphere.*

S.Akbulut independently proved the same result in [2].

### 3 Handle decomposition of $\mathcal{S}_{m,n,m',n'}$ .

#### 3.1 The diagram of $X_{m,n}$ .

**Lemma 2.** *A handle decomposition of  $X_{m,n}$  is Figure 4.*

**Proof.** First the picture of  $T^4$  is the left of Figure 5. Recall that  $T^3$  is obtained from 0-surgery along the Borromean ring. Since  $T_0^2 \times T_0^2$  is obtained by removing  $D_0^2 \times T^2 \subset T^2 \times D^2$  and  $T_0^2 \times D_0^2 \subset T_0^2 \times T^2$ , the diagram is the right of Figure 5. Since  $(m/1)$  and  $(n/1)$ -log transformation correspond to the  $(0, 1/m, 1/n)$  surgery over the Borromean ring, we get Figure 4 as a diagram of  $X_{m,n}$ .  $\square$

The 3 directions of the right in Figure 5 represent  $S_\alpha^1, S_\beta^1, S_\gamma^1$  generating circles above.

#### 3.2 Upside-down of $X_{m',n'}$ .

Next we perform the upside down of the manifolds  $X_{m',n'}$ . The right four 2-handles in Figure 4 which are along two components of Borromean ring and the two meridians are, as each runs through the adjacent 1-handles once, canceled each other. In addition the top four 2-handles are isotopic to trivial unknots on the boundary and they are canceled out with four 3-handles. Then attaching dual 2-handles are the meridian for the bottom four 2-handles as in Figure 4. Here we present the dual 2-handle by red lines. Then by handle sliding we get the diagram Figure 7.

In addition several handle slides give Figure 8 and 9. Here replacing the two handles as in Figure 11 we get Figure 10. Using the notation and isotopy we get Figure 12 and keep track of the red two handles by the symmetry that exchanges the pair of link  $(a, b)$  to  $(c, d)$ , hence we get Figure 13. Keeping track of the diagram by the converse motion (Figure 12-10-9-8-7-6-4) from the diagram in the form, we get Figure 17.

### 4 Handle calculus of $\mathcal{S}_{m,n,m',n'}$ .

**Proposition 1.** *Each of the manifolds  $\mathcal{S}_{m,n,m',n'}$  admits a handle decomposition without 1-handles. In addition the handle decomposition has 4 2-handles.*

**Proof.** To prove this lemma, we will find eight 1,2-canceling pairs. Any canceled pair is drawn by dotted line. Here the only 1-handle goes on drawing as a ball description. First we take 4 pairs below as Figure 18. In addition we take 4 pairs below as Figure 19 and 20. Hence we get a handle decomposition

$$\mathcal{S}_{m,n,m',n'} = D^4 \cup^8 2\text{-handles} \cup^8 3\text{-handles} \cup 4\text{-handle}$$

Then, sliding among several canceled handles, we get the figure that the attaching circles  $\alpha, \beta, \chi, \delta$  are isotopic to the unlink in  $\partial D^4$  hence these are canceled out with 4 3-handles in the manifold.

We get a handle decomposition

$$\mathcal{S}_{m,n,m',n'} = D^4 \cup^4 2\text{-handles} \cup^4 3\text{-handles} \cup 4\text{-handle}. \quad (1)$$

□

Thus we get Figure 22 as a diagram of  $\mathcal{S}_{m,n,m',n'}$ . We put the framed link in  $\partial D^4$  of the 2-handles as  $\mathcal{F}_{m,n,m',n'}$ .

Here four attaching circles (red lines,  $\epsilon, \phi, \gamma, \eta$ ) represent the 2-handles in (1).

Next we show the following.

**Lemma 3.** *The 0-framed 2-handles  $\mathcal{F}_{m,n,m',n'}$  are, after several handle slides, isotopic to  $\mathcal{F}_{m,0,m',n'}$ . Furthermore two  $\gamma, \phi$  of them are separated as 2-component unlink after handle slidings.*

**Proof.** Replacing dots of 1-handle to 0-framed 2-handles and sliding handles, we get Figure 23. Two handle slides give Figure 24. Sliding handle as indicated in the figure, we get Figure 25, and by isotopy we get Figure 26. Turning the diagram in the direction of the arrow in Figure 26  $n'$  times, we obtain Figure 27. Removing bottom canceling 1,2-handle pair, we get Figure 28. Sliding central 2-handle, we get Figure 29. The curve  $\gamma$  in Figure 29 is untied by several handle slidings to get a separated 2-handle as in Figure 30. At this time the  $n$  and  $-n$  boxes are untied by rotating (Figure 31). Sliding and canceling handles, we get Figure 32 and 33. In the form we can untie  $\epsilon$  by a handle slide as in Figure 34. Iterating this process, we get Figure 35. □

#### 4.1 Nash's manifolds as a torus surgery.

In the subsection we show that each of Nash's manifold is constructed by a log transformation along a single torus.

**Proposition 2.** *For any  $m, n, m', n'$  we have*

$$\mathcal{S}_{m,n,0,n'} \cong \mathcal{S}_{m,n,m',0} \cong S^4.$$

**Proof.** Putting  $m' = 0$ , we have Figure 36. The resulting manifold is the surgering of  $S^3 \times S^1$  along  $\{\text{pt}\} \times S^1$  framing  $n'$ . Namely the manifold has the same as Figure 39. This is diffeomorphic to  $S^4$ . The manifold  $\mathcal{S}_{m,n,m',0}$  is also diffeomorphic to  $S^4$  in the similar way. □

As a corollary we have the following.

**Corollary 1.**  *$\mathcal{S}_{m,n,m',n'}$  are given by one log transformation along a torus.*

Now we are in a position to prove the main theorem.

## 4.2 Proof of Theorem 1.

By Lemma 3 the handle decomposition of  $\mathcal{S}_{m,n,m',n'}$  is  $D^4$  and the same framed link as  $\mathcal{F}_{m,0,m',n'}$  and four 3-handles and a 4-handle. Namely  $\mathcal{S}_{m,n,m',n'}$  is the same handle decomposition as  $\mathcal{S}_{m,0,m',n'}$ . In particular we have  $\mathcal{S}_{m,n,m',n'} \cong \mathcal{S}_{m,0,m',n'}$ . From the Lemma 1 and Proposition 2 we have  $\mathcal{S}_{m,n,m',n'} \cong S^4$ .  $\square$

**Corollary 2.** *The diagram Figure 1 is framed link presentation of  $\#^2 S^2 \times S^1$ .*

**Proof.** Figure 35 gives a handle decomposition of  $S^4$ :

$$D^4 \cup^2 2\text{-handles} \cup^2 3\text{-handles} \cup 4\text{-handle}.$$

Therefore the boundary  $\partial(D^4 \cup^2 2\text{-handles})$  is  $\#^2 S^2 \times S^1$ .  $\square$

This corollary implies  $\epsilon$  and  $\eta$  in Figure 1 are candidates of counterexample of generalized Property R conjecture.

## References

- [1] S. AKBULUT, *Cappell-Shaneson homotopy spheres are standard*, Ann. of Math. 149 (1999), 497-510. Zbl
- [2] S. AKBULUT, *Nash homotopy spheres are standard*, arXiv:1102.2683v1
- [3] S. AKBULUT AND R. KIRBY, *Apotential smooth counterexample in dimension 4 to the Poincaré conjecture, the Schoenflies conjecture, and the Andrews-Curtis conjecture*, Topology 24 (1985), no. 4, 375-390.
- [4] S.E.CAPPELL, AND J.L.SHANESON, *There exist inequivalent knots with the same complement*, Ann. of Math. (2) 103 (1976), no. 2, 349-353.
- [5] R. FINTUSHEL AND STERN, *Pinwheels and nullhomologous surgery on 4-manifolds with  $b^+ = 1$*  arXiv:maht/1004.3049v1
- [6] D. GABAI, *Foliations and the topology of 3-manifolds. III*, J. Differential Geom. 26 (1987), 479-536.
- [7] R. GOMPF, *On Cappell-Shaneson 4-spheres*, Topology Appl., Volume 38, Issue 2, 28 February 1991, Pages 123-136
- [8] R. GOMPF, *Killing the Akbulut-Kirby 4-sphere, with relevance to the Andrews-Curtis and Schoenflies problems*, Topology 30 (1991), no. 1, 97-115.
- [9] R. GOMPF, *More Cappell-Shaneson spheres are standard*, Algebr. Geom. Topol. 10 (2010), no. 3, 1665-1681
- [10] M. FREEDMAN, R. GOMPF, S.MORRISON, AND K. WALKER *Man and machine thinking about the smooth 4-dimensional Poincare conjecture*, Quantum Topol. 1 (2010), no. 2, 171-208
- [11] D. NASH, *New Homotopy 4-Spheres* arXiv:maht/1101.2981v1
- [12] M. SHARLEMANN AND A. THOMPSON, *Fibered knots and Property 2R* arXiv:maht/0901.2319v1

Motoo Tange  
 Research Institute for Mathematical Sciences,  
 Kyoto University,  
 Kyoto 606-8502, Japan.  
 tange@kurims.kyoto-u.ac.jp

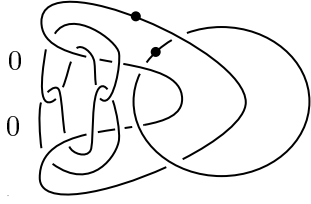


Figure 2:  $A$ .

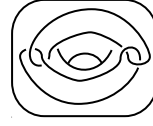


Figure 3: Bing tori.

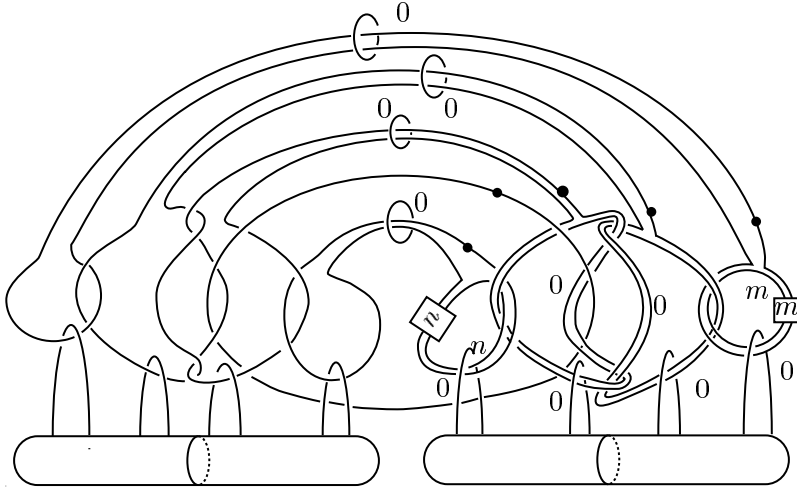


Figure 4:  $X_{m,n}$

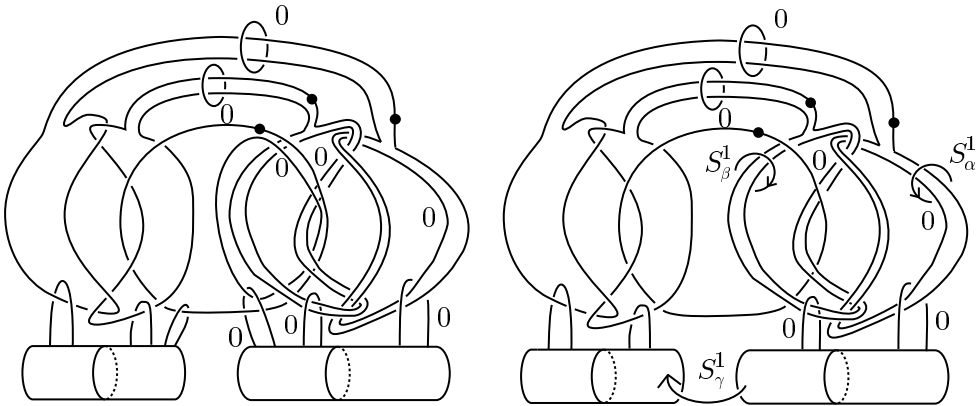


Figure 5:  $T^4$  and  $T_0^2 \times T_0^2$ .

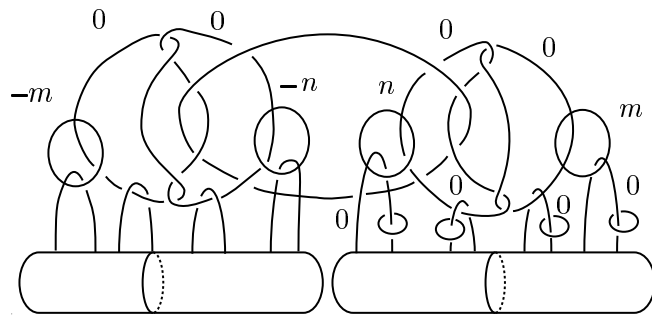


Figure 6:

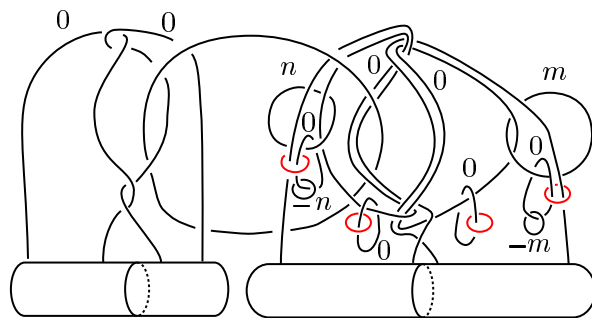


Figure 7:

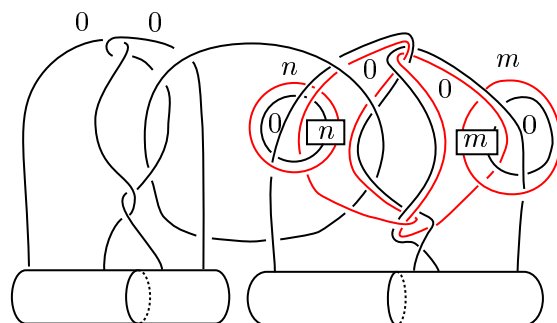


Figure 8:

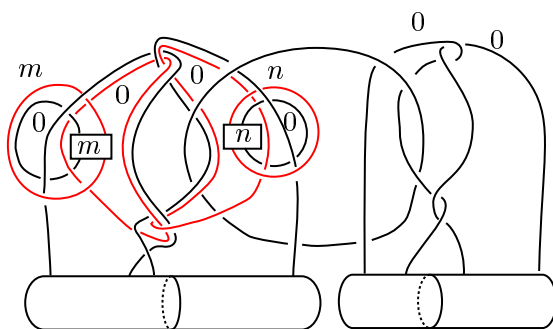


Figure 9:

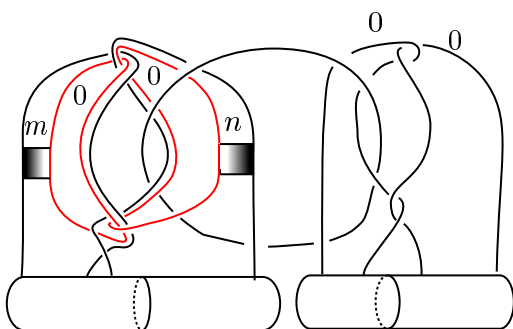


Figure 10:

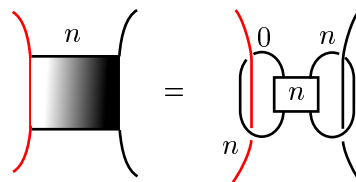


Figure 11:

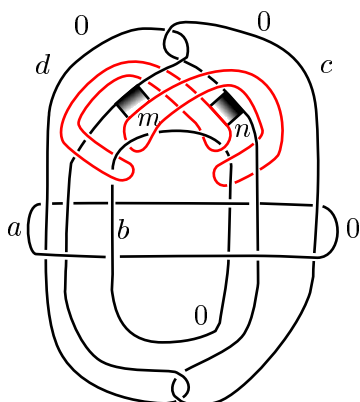


Figure 12:

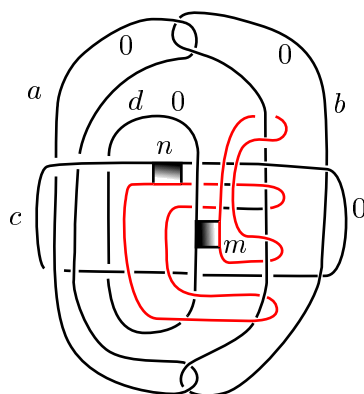


Figure 13:



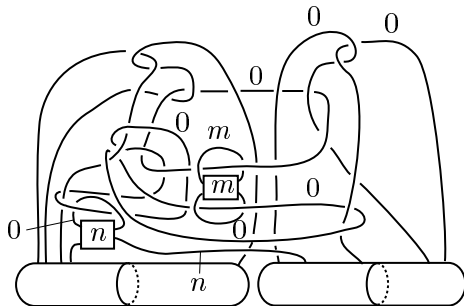


Figure 14:

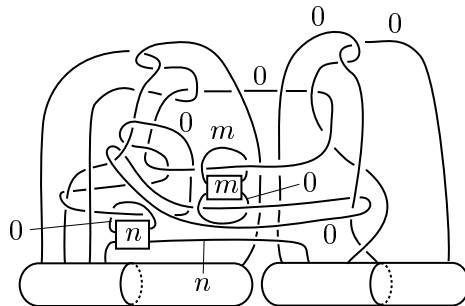


Figure 15:

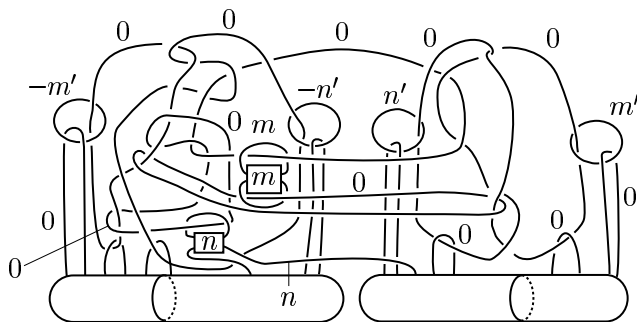


Figure 16:  $X_{m,n}$

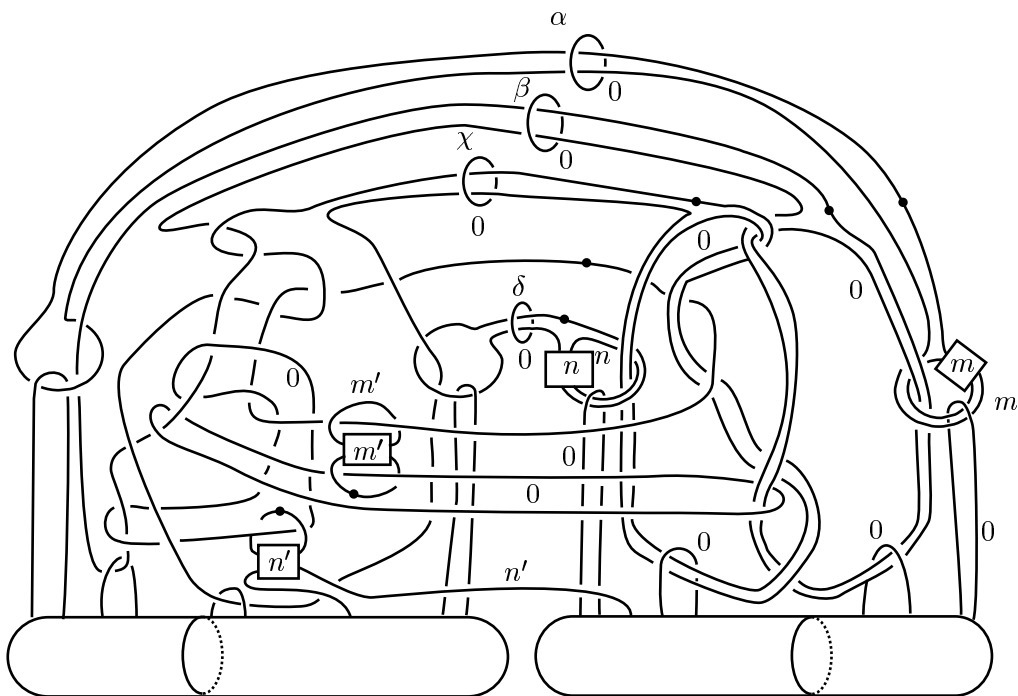


Figure 17:  $\mathcal{S}_{m,n,m',n'}$ .

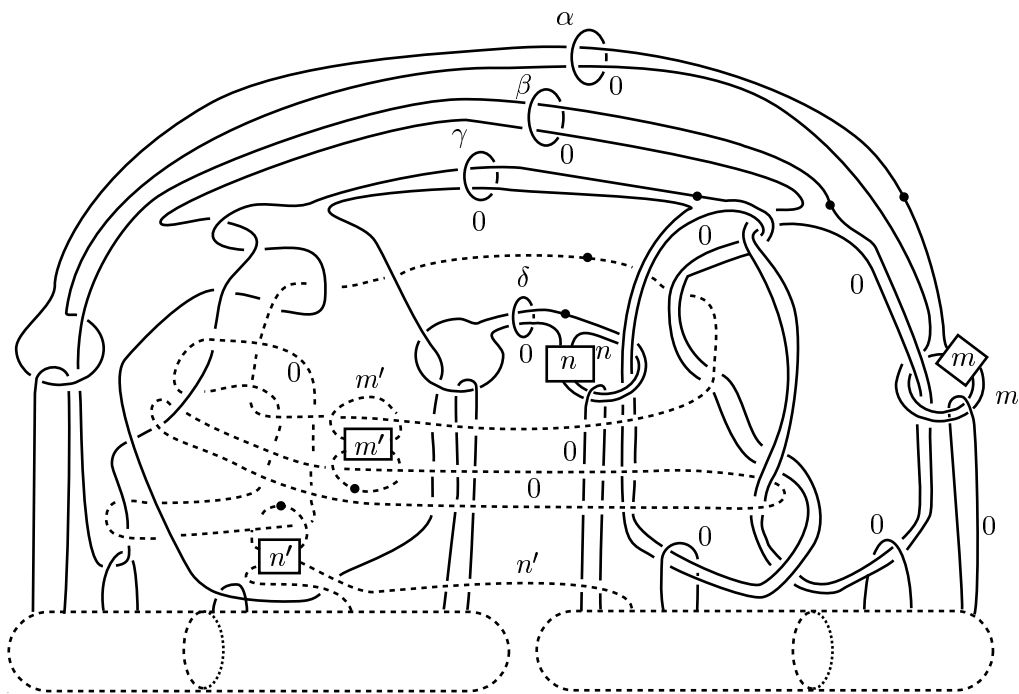


Figure 18:  $\mathcal{S}_{m,n,m',n'}$  canceled.

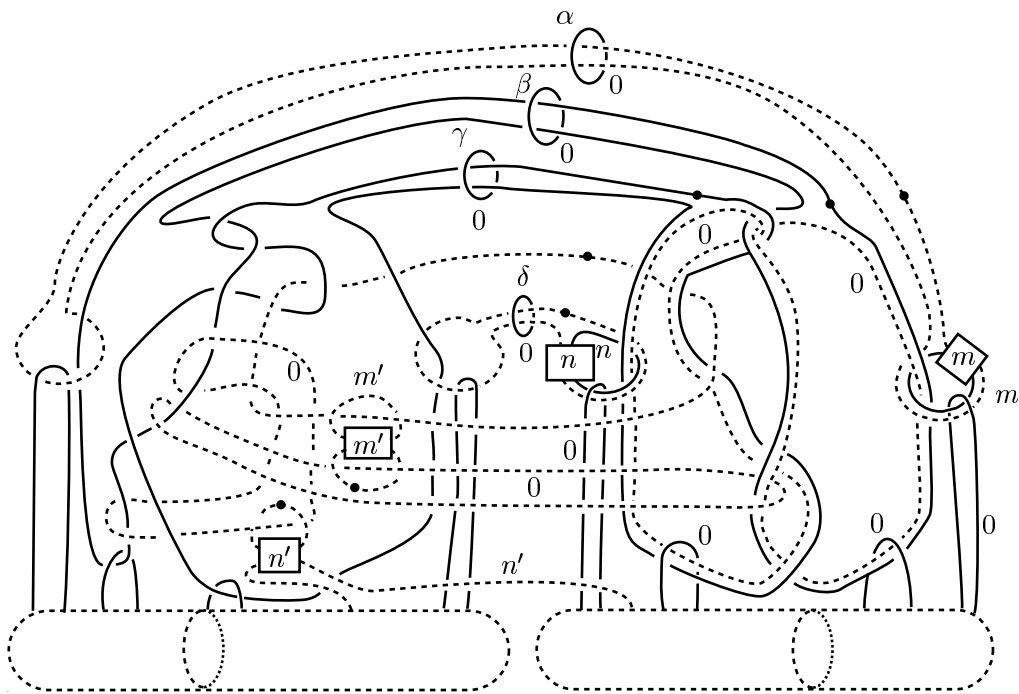


Figure 19:  $\mathcal{S}_{m,n,m',n'}$  canceled.

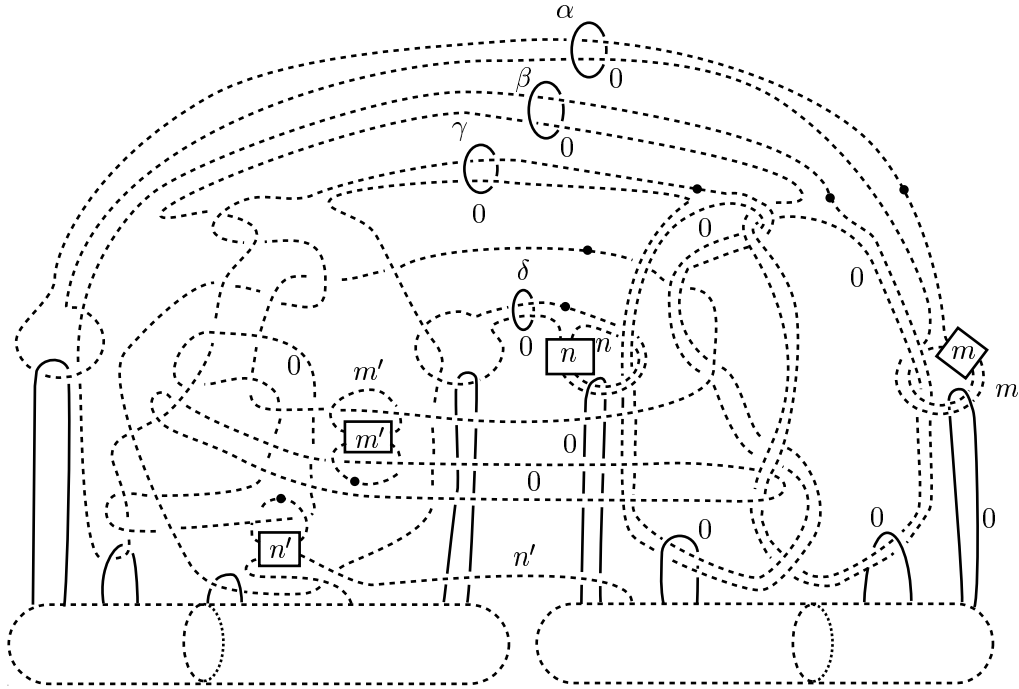


Figure 20:  $\mathcal{S}_{m,n,m',n'}$  canceled.

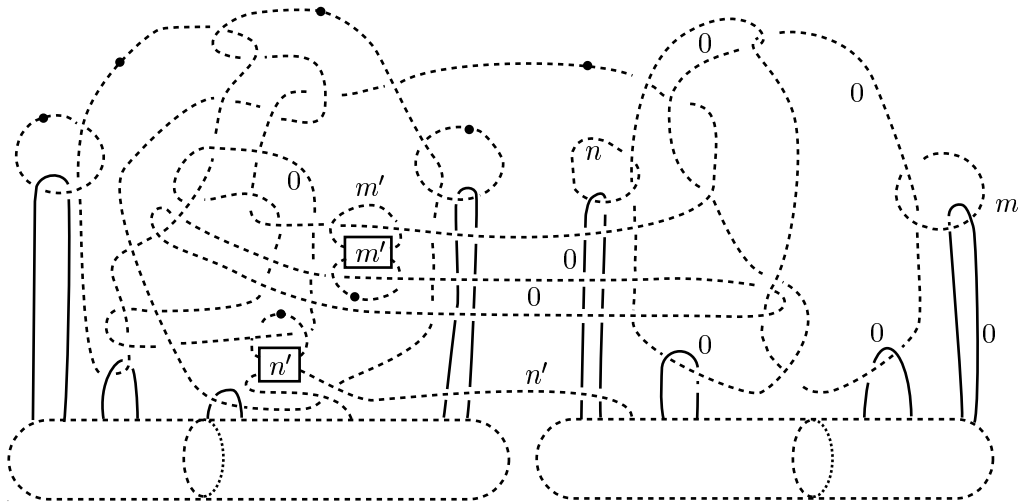


Figure 21:  $\mathcal{S}_{m,n,m',n'}$  canceled.

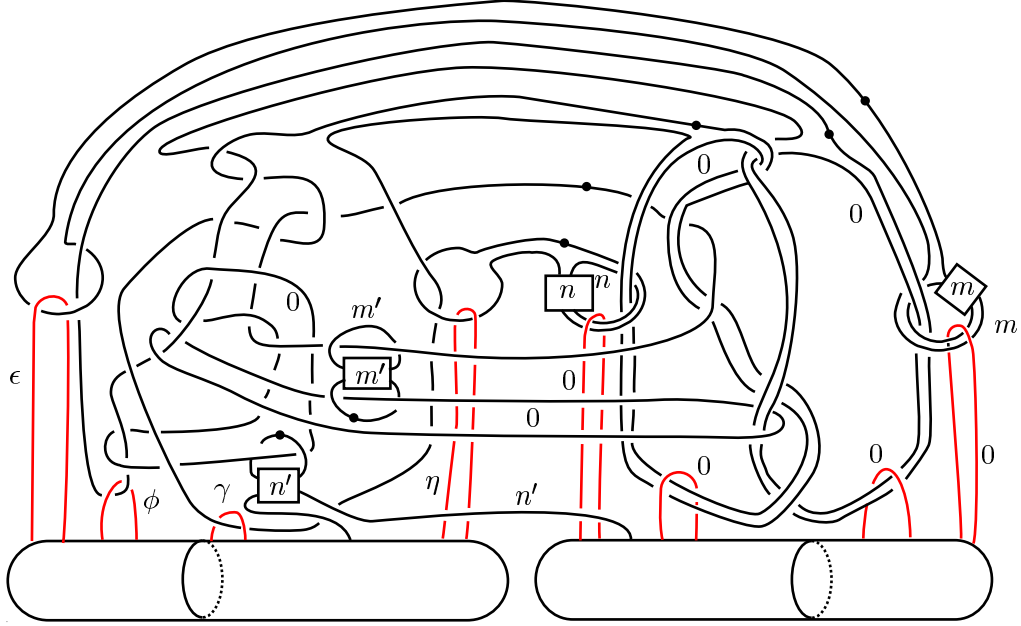


Figure 22:  $\mathcal{S}_{m,n,m',n'}$  canceled.

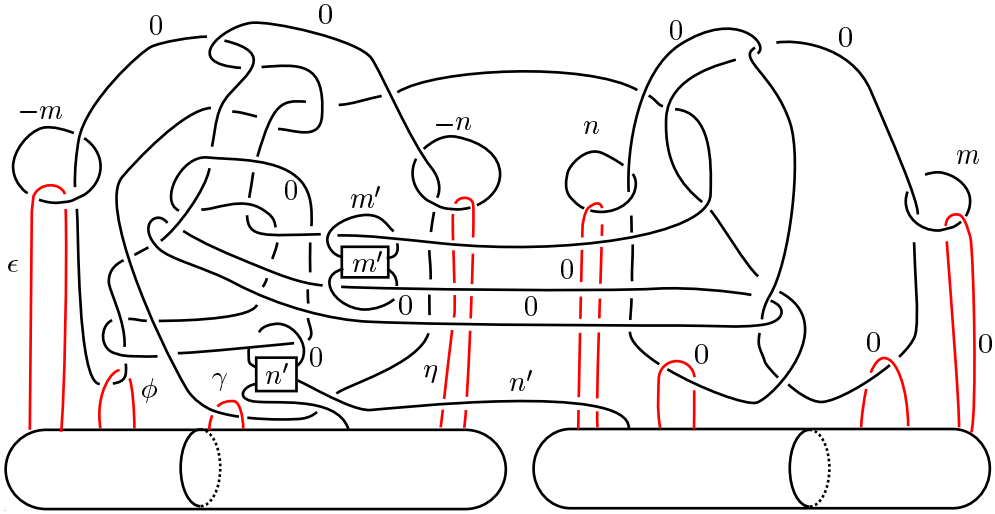


Figure 23: 2-handles over  $\partial D^4$ .

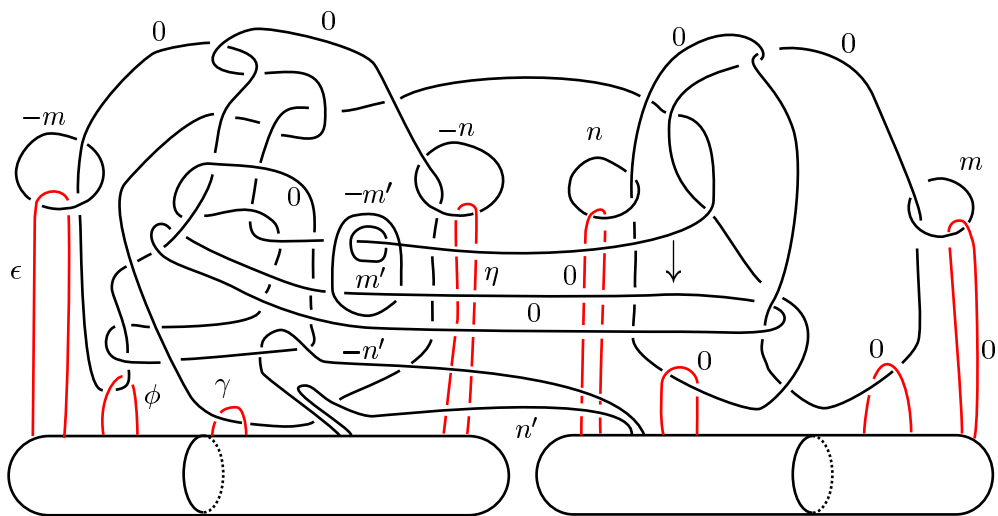


Figure 24: 2-handles over  $\partial D^4$ .

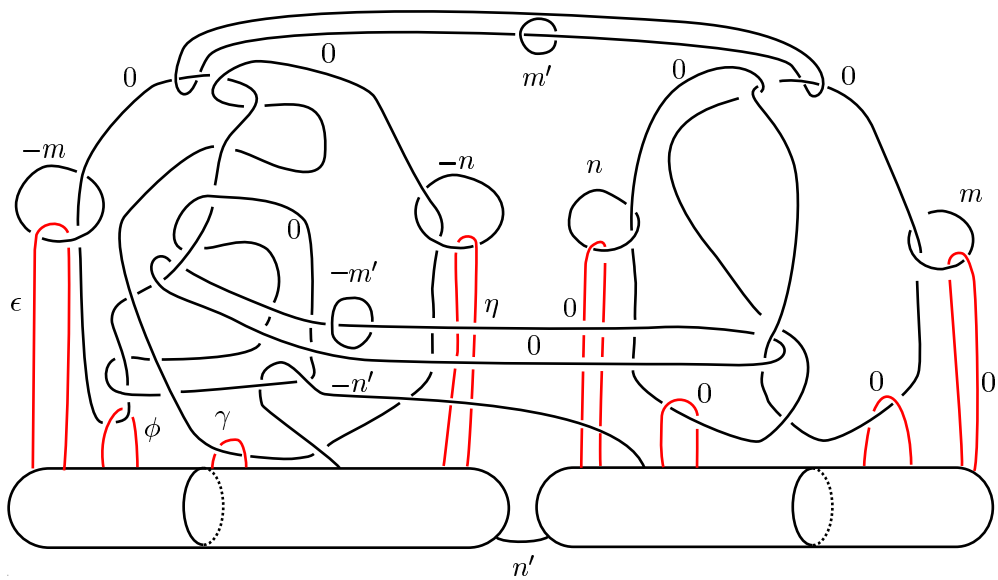


Figure 25: 2-handles over  $\partial D^4$ .

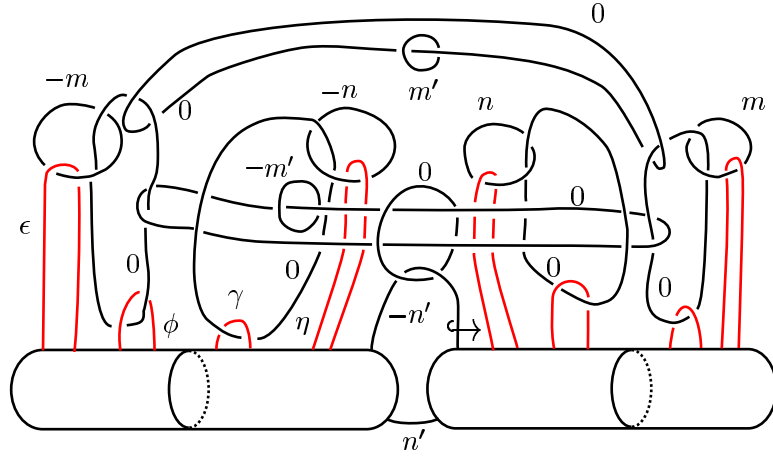


Figure 26: 2-handles over  $\partial D^4$ .

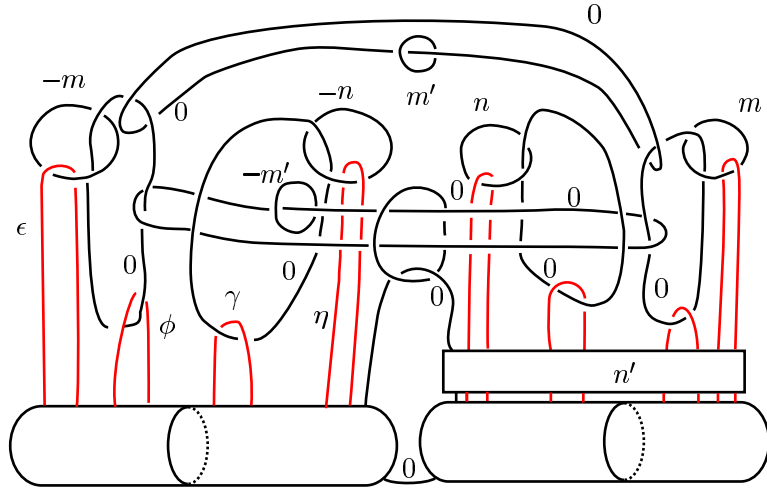


Figure 27: 2-handles over  $\partial D^4$ .



0

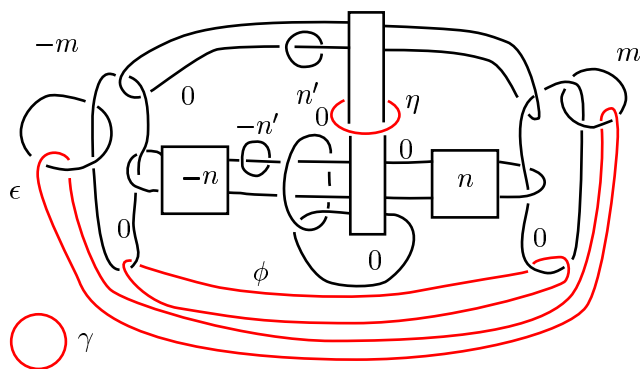


Figure 30: 2-handles over  $\partial D^4$ .

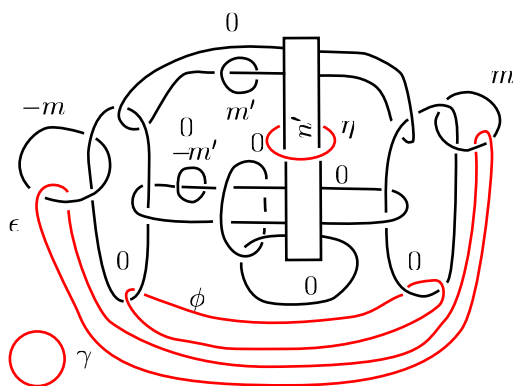


Figure 31: 2-handles over  $\partial D^4$ .

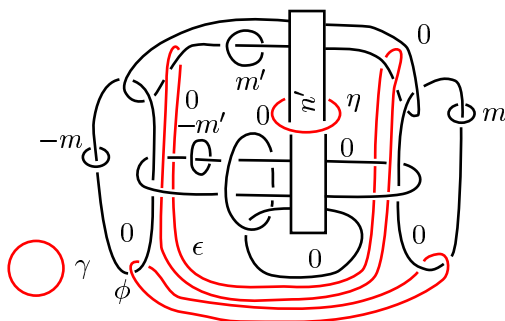


Figure 32: 2-handles over  $\partial D^4$ .



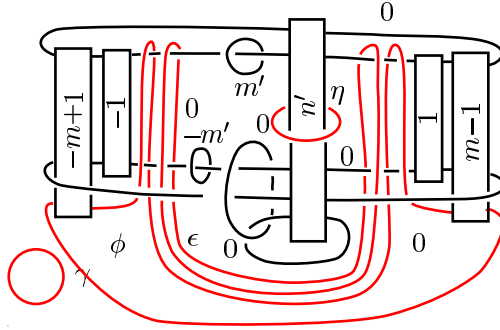


Figure 33: 2-handles over  $\partial D^4$ .

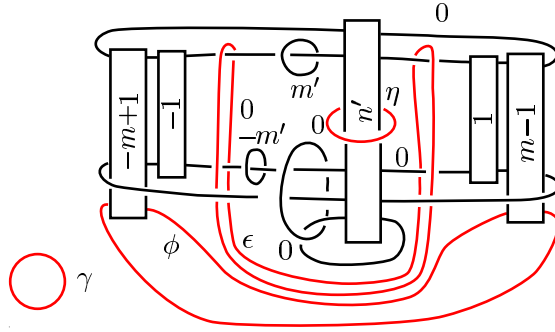


Figure 34: 2-handles over  $\partial D^4$ .

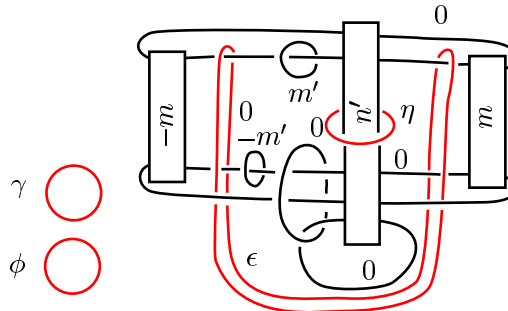


Figure 35: 2-handles over  $\partial D^4$ .

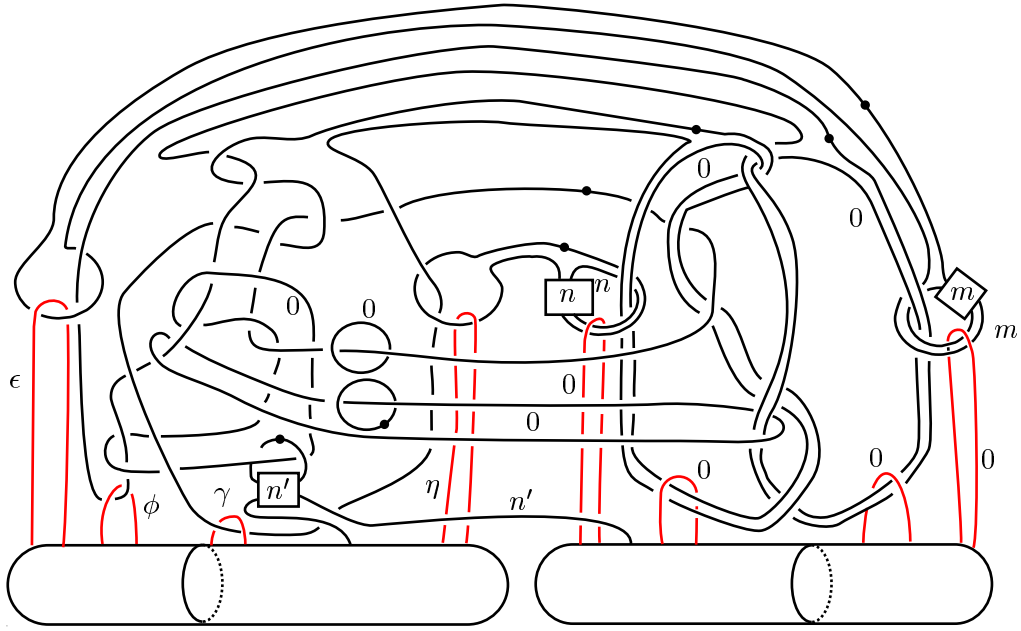


Figure 36:  $\mathcal{S}_{m,n,0,n'}$

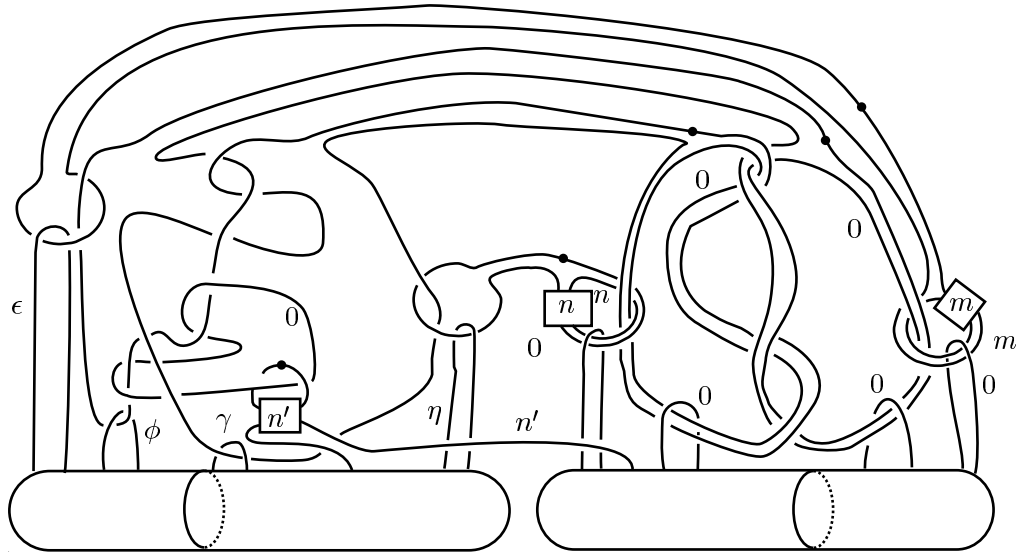


Figure 37:  $\mathcal{S}_{m,n,0,n'}$

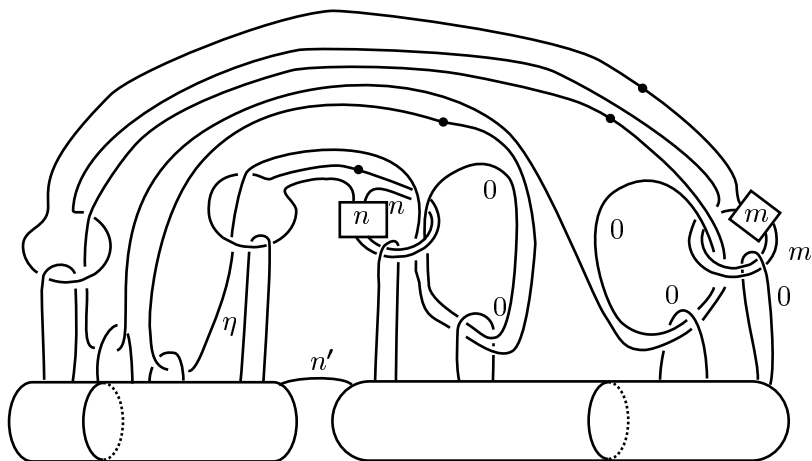


Figure 38:  $\mathcal{S}_{m,n,0,n'}$

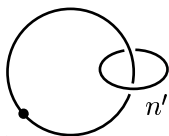


Figure 39:  $\mathcal{S}_{m,n,0,n'}$